$$\left|\frac{1}{u_y}(u_{yx}u_y - u_x u_{yy}) + \frac{U_x}{u_y U}(uu_{yy} - u_y^2)\right| \leqslant K_9 (U - u)\sigma$$

$$-K_{10}(U - u)\sigma \leqslant \frac{1}{u_y}(u_{yt}u_y - u_t u_{yy}) + \frac{U_t}{u_y U}(uu_{yy} - u_y^2) \leqslant K_{11} x (U - u)\sigma$$

are satisfied. In the above inequalities $\sigma = [-\ln \mu\ (1 - u / U)]^{1/2}$, K_i and μ are certain positive constants, $0 < \mu < 1$. and $X_2 > 0$ depends on U, r and v_0.

This theorem is the corollary of Theorem 1.

We note in conclusion that the stipulations and the input data of problem (1), (2) formulated in Lemma 4 and Theorems 1 and 2 are somewhat less stringent than the limitations imposed in [1]. The analysis presented here has to a certain extent improved the results obtained in [1] and made it possible to prove the theorem of existence of solution of the Cauchy problem for the requirements with respect to external flow, as specified in Theorems 1 and 2.

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ON THE PLANE-PARALLEL SYMMETRIC BOUNDARY LAYER GENERATED BY SUDDEN MOTION

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The formation of a boundary layer over a body which suddenly begins to move in a stationary incompressible fluid is analyzed. Proof is given of the existence and uniqueness under certain conditions of solution of the related boundary value problem defined by the system of Prandtl's equations in a certain time interval $0 \leqslant t \leqslant T$ and over the whole of the streamlined body. This problem was also considered by Blasius [1] who had proposed to solve it by expanding the stream function into an asymptotic series in powers of time, and had given the first two terms of this expansion in their explicit form. A brief account of these results and the mathematical formulation of the problem appear in [2, 3]. The problem of boundary layer development under conditions of gradual acceleration was

examined in [4, 5].

The problem of boundary layer formation in a plane-parallel symmetric flow past a body by sudden motion reduces to the analysis of the following system of equations:

$$u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \qquad u_x + v_y = 0 \tag{1}$$

in region $D_T\ \{0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant X,\ 0 \leqslant y < \infty\}$ with conditions

$$u|_{t=0} = U(0,\ x), \qquad u|_{y=0} = 0 \quad \text{for } t > 0, \qquad u|_{x=0} = 0$$
$$v|_{y=0} = v_0(t,\ x), \qquad u \to U(t,\ x) \quad \text{for } y \to \infty \tag{2}$$

where u and v are velocity components, $U\ (t,\ x)$ is the longitudinal velocity component of the external stream, $U\ (t,\ 0) = 0,\ \ U\ (t,\ x) > 0$ for $x > 0$ and $-p_x = U_t + UU_x$. Substituting the independent variables

$$\tau = \sqrt{t}, \quad \xi = x, \quad \eta = u/U \tag{3}$$

and introducing the new unknown function

$$w(\tau,\ \xi,\ \eta) = \sqrt{t} u_y / U \tag{4}$$

we reduce system (1) with conditions (2) to a single equation

$$\nu w^2 w_{\eta\eta} - {}^1/_2 \tau w_\tau - \tau^2 \eta U w_\xi + A w_\eta + B w = 0, \tag{5}$$

in region $\Omega\ \{0 \leqslant \tau \leqslant \sqrt{T},\ \ 0 \leqslant \xi \leqslant X,\ \ 0 \leqslant \eta \leqslant 1\}$ with boundary conditions

$$w|_{\eta=1} = 0 \quad (\nu w w_\eta - \tau v_0 w + C)|_{\eta=0} = 0 \tag{6}$$

$$A = \tau^2(\eta^2 - 1) U_x + \tau^2(\eta - 1)\frac{U_t}{U}, \quad B = -\eta\tau^2 U_x - \tau^2 \frac{U_t}{U} + \frac{1}{2}$$

$$C = \tau^2 U_x + \tau^2 \frac{U_t}{U}$$

With the use of the method of straight lines [6] we shall prove the existence of solution of the problem (5), (6) and establish certain estimates for w. Using transformations (3) and (4) we obtain formulas which are required for determining u and v of problem (1) with conditions (2).

We denote $f(kh,\ lh,\ \eta)$, where $h = \text{const} > 0$, in the arbitrary function $f(\tau,\ \xi,\ \eta)$ by $f^{k,\,l}(\eta)$, and substitute for Eq. (5) with conditions (6) the system of ordinary differential equations

$$L^{k,l}(w) \equiv \nu (w^{k,l})^2 w^{k,l}_{\eta\eta} - \frac{kh}{2}\ \frac{w^{k,l} - w^{k-1,l}}{h} -$$
$$-(\eta + \lambda h)(kh)^2 U^{k,l} \frac{w^{k,l} - w^{k,l-1}}{h} + A^{k,l} w^{k,l}_\eta + B^{k,l} w^{k,l} = 0$$
$$k = 0,\ 1,\ \ldots,\ \left[\frac{\sqrt{T}}{h}\right], \qquad l = 0,\ 1,\ \ldots,\left[\frac{X}{h}\right] \tag{7}$$

along the segment $0 \leqslant \eta \leqslant 1$ with conditions

$$w^{k,l}(1) = 0, \quad l^{k,l}(w) \equiv (\nu w^{k,l} w^{k,l}_\eta - khv_0^{k,l} w^{k,l} + C^{k,l})|_{\eta=0} = 0 \tag{8}$$

In Eqs. (7) λ is a sufficiently great positive constant independent of h.

Henceforth C_i and M_i will denote positive constants independent of h and $\sigma_\gamma = \sqrt{-\ln \gamma(1-\eta)}$, where $0 < \gamma < 1$ and $0 \leqslant \eta < 1$.

Lemma. Let us assume that

$$U_x + U_t/U > 0,\ U_x(0,\ 0) > 0,\ v_0 \leqslant M_1 \tau^{1+\varepsilon},\ \varepsilon > 0,\ |v_{0x}| \leqslant M_2 \tau,$$

and functions v_0, U, U_x, U_t/U, U_{xx}, $U_{x\tau}$, $(U_t/U)_x$ are bounded, and $(U_t/U) \geqslant -M_3U$ in a certain neighborhood of $x=0$. Then, for $0 \leqslant lh \leqslant X$ and $0 \leqslant kh \leqslant \sqrt{T}$, where T depends on U, v_0 and ν, the system of Eqs. (7) with conditions (8) has the solution $w^{k,l}(\eta)$ which is continuous for $0 \leqslant \eta \leqslant 1$, has a continuous third-order derivative for $0 \leqslant \eta < 1$, and satisfies the inequalities

$$C_1(1-\eta)\sigma_\mu \leqslant w^{k,l} \leqslant C_2(1-\eta)\sigma_{\mu_1}, \qquad 0<\mu_1<\frac{1}{\sqrt{e}}<\mu<1 \tag{9}$$

Furthermore, the following estimates are valid:

$$|(w^{k,l}-w^{k,l-1})/h| \leqslant C_3(1-\eta)\sigma_{\mu_1} \tag{10}$$

$$|(w^{k,l}-w^{k-1,l})/h| \leqslant C_4(1-\eta)\sigma_{\mu_1} \tag{11}$$

$$-C_5\sigma_{\mu_1} \leqslant w_\eta^{k,l} \leqslant C_6\Phi(\eta) \tag{12}$$

where $\Phi(\eta) = -\eta^2$ for $0 \leqslant \eta < 1-\delta$ and $\Phi(\eta) = -\sigma_\mu$ for $(1-\delta) \leqslant \eta < 1$ and

$$|w^{k,l}w^{k,l}_{\eta\eta}| \leqslant C_7, \qquad w^{k,l}w^{k,l}_{\eta\eta} < -1/(4\nu) \tag{13}$$

The existence of functions $w^{0,l}(\eta)$ $(l=0, 1, \ldots, [X/h])$ and of estimate (9) for $w^{0,l}$ readily follow from Eqs. (7) for $k=0$, while for $k \geqslant 1$ the solution $w^{k,l}$ of the system of Eqs. (7) with conditions (8) is obtained as the limit for $\varepsilon_1 \to 0$ of solutions of the system

$$L^{k,l}(w) + \varepsilon_1 w^{k,l}_{\eta\eta} = 0,\ \varepsilon_1 > 0,\ 1 \leqslant k \leqslant [\sqrt{T}/h],\ 0 \leqslant l \leqslant [X/h]$$

with conditions (8). The proof of this and of estimates (9) is similar to that of Lemmas 3 and 4 in [6]. To prove the estimates (10) – (12) we use the equations and boundary conditions which are satisfied by the following quantities:

$$r^{k,l} = (w^{k,l}-w^{k,l-1})/h,\ \rho^{k,l} = (w^{k,l}-w^{k-1,l})/h,\ z^{k,l} = w^{k,l}_\eta$$

For example, function $r^{k,l}$ satisfies equation

$$R^{k,l}(r^{k,l}) = [L^{k,l}(w) - L^{k,l-1}(w)]/h = 0$$

and the boundary conditions

$$r^{k,l}(1) = 0,\ \lambda^{k,l}(r^{k,l}) = [l^{k,l}(w) - l^{k,l-1}(w)]/h = 0 \quad \text{for } \eta = 0$$

Now, assuming that the inequalities (10) – (12) are satisfied for $w^{k',l'}(\eta)$ when $k' \leqslant (k-1)$, $k'=k$ and $l' \leqslant l-1$, we can prove that for sufficiently small T these inequalities are also valid for $k'=k$ and $l'=l$. Proof of this is similar to that of Lemma 9 in [6]. Inequalities (13) are the consequence of estimates (10) – (12) and are derived from Eqs. (7).

Theorem 1. Let the conditions of the lemma be fulfilled. Then in the region $\Omega\ \{0 \leqslant \tau \leqslant \sqrt{T},\ 0 \leqslant \xi \leqslant X,\ 0 \leqslant \eta \leqslant 1\}$ for any X and a certain T dependent on U, v_0 and ν there exists a solution of problem (5), (6) which has the following properties: w is continuous in Ω; $C_1(1-\eta)\sigma_\mu \leqslant w \leqslant C_2(1-\eta)\sigma_{\mu_1}$; function w_η is continuous with respect to η for $\eta < 1$; $-C_5\sigma_{\mu_1} \leqslant w_\eta \leqslant C_6\Phi(\eta)$, where Φ is the function defined in the lemma; $|w_\xi| \leqslant C_3(1-\eta)\sigma_{\mu_1}$; $|w_\tau| \leqslant C_4(1-\eta)\sigma_{\mu_1}$, $ww_{\eta\eta}$ is bounded, and $ww_{\eta\eta} < -1/(4\nu)$. This function satisfies almost everywhere Eq. (5) and conditions (6) in space Ω. These properties are unique to the solution of problem (5), (6).

The existence of solution w of problem (5), (6) which has the above properties follows from the solution $w^{k,l}$ of problem (7), (8) and estimates (9) – (13). To prove the uniqueness of the solution of problem (5), (6) we examine the remainder $w_1 - w_2 = W$ of two solutions of this problem. Function W satisfies equation

$$P(W) \equiv \nu w_1 W_{\eta\eta} - \frac{\tau}{2}\frac{W_\tau}{w_1} - \tau^2\eta U\frac{W_\xi}{w_1} + B\frac{W}{w_1} + A\frac{W_\eta}{w_1} + \nu w_{2\eta\eta}\frac{w_1 + w_2}{w_1}W = 0$$

with boundary conditions

$$W\Big|_{\eta=1} = 0, \qquad \left(\nu W_\eta - C\frac{W}{w_1 w_2}\right)\Big|_{\eta=0} = 0$$

Let us consider

$$\int_\Omega P(W)\, We^{-\alpha\tau}d\tau d\xi d\eta = 0, \ \alpha = \text{const} > 0 \tag{14}$$

Integrating by parts the derived integrals, from (14) we obtain an inequality of the form

$$\int_\Omega K(w_1, w_2, \tau, \xi, \eta, \alpha)\, W^2 e^{-\alpha\tau}d\tau d\xi d\eta \geqslant 0 \tag{15}$$

where for sufficiently great α function $K(w_1, w_2, \tau, \xi, \eta, \alpha) < 0$. Hence it follows from (15) that $W^2 \equiv 0$, i.e., $w_1 \equiv w_2$.

Theorem 2. Let us assume that functions U_x, U_{xx}, U_t / U, $(U_t / U)_x$, $\sqrt{t}\, U_{xt}$, $\sqrt{t}\,(U_t/U)_t$, v_0 and $\sqrt{t}\, v_{0t}$ are bounded, $U_x(0, 0) > 0$, $(U_x + U_t / U) > 0$ and $-M_3 U \leqslant U_t / U$ for small x. Let also $v_0 \leqslant Mt^{1/2+\varepsilon}$, $\varepsilon > 0$, and $|v_{0x}| \leqslant M_2 t^{1/2}$.Then in region D_T, where T is dependent on U, v_0 and ν there exists a unique solution for u and v of problem (1), (2), which has the following properties: u/U is continuous for $t > 0$: $\sqrt{t}u_y / U$ is bounded and continuous; $u \to U$, when $y \to \infty$; u and v satisfy conditions (2), and u_y, u_x, u_{yy}, u_t and v_y are bounded and continuous for $t > 0$. The equations of system (1) are satisfied almost everywhere in D_T. Furthermore, the following inequalities are valid:

$$C_1\left(1 - \frac{u}{U}\right)\sigma_\mu\left(\frac{u}{U}\right) \leqslant \frac{\sqrt{t}\,u_y}{U} \leqslant C_2\left(1 - \frac{u}{U}\right)\sigma_{\mu_1}\left(\frac{u}{U}\right) \tag{16}$$

$$U(t, x)\exp\left(-\frac{C_2^2 y^2}{4t} - \frac{C_2 y\sqrt{-\ln\mu_1}}{\sqrt{t}}\right) \leqslant U(t, x) - u \leqslant \leqslant U(t, x)\exp\left(-\frac{C_1^2 y^2}{4t} - \frac{C_1 y\sqrt{-\ln\mu}}{\sqrt{t}}\right) \tag{17}$$

The proof of this theorem is similar to that of Theorem 2 in [6] and in [4].

Estimates (17) define the rate at which $u(t, x, y)$ tend to $U(t, x)$ when $y \to \infty$ and, also, the behavior of u at small t and fixed x and y, which is associated with the rate of the boundary layer build-up. Inequalities (17) also imply that function u satisfies conditions (2). Estimates of the form (17) are also valid for the sum of the first terms of the expansion of function u in powers of time determined by Blasius in [1].

It follows from (3) and (4) that the approximate solution $u((kh)^2, lh, y)$ of problem (1), (2) can be derived with the use of functions $w^{k,l}$ by formula

$$\frac{y}{\sqrt{kh}} = \int_0^\eta \frac{ds}{w^{k,l}(s)}, \qquad \eta = \frac{u((kh)^2, lh, y)}{U((kh)^2, lh)}$$

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DIFFUSION ON A PARTICLE IN THE SHEAR FLOW OF A VISCOUS FLUID. APPROXIMATION OF THE DIFFUSION BOUNDARY LAYER

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The steady convective diffusion on the surface of a particle of a substance dissolved in a uniform shear flow of viscous flow is considered. The problem of diffusion on a solid sphere and a spherical drop is solved in the approximation of the diffusion boundary layer.
Determination of diffusion afflux of a substance (or heat) on the surface of a moving particle is one of the fundamental problems of physicochemical hydrodynamics related to the theory of combustion, chemical reactors, in particular those with suspended layers, to the theory of coagulation and flocculation of disperse systems, deposition of aerosols, and in numerous other applications.
The analytical solutions obtained so far relate only to straight, uniform at infinity, laminar flows past particles at low Reynolds numbers [1 - 6].
Here an approximate analytical expression is derived for the diffusing stream of a substance on the surface of a spherical particle in a uniform laminar shear flow. Stokes' approximation derived in [7] is used for determining the shear flow field. It is assumed that the Péclet number is considerable so that the equation of convective diffusion can be expressed in terms of boundary layer approximation.

1. Statement of problem. Let us consider a spherical particle carried along by a stream of viscous incompressible fluid in a steady uniform shear flow. In an